TRINITY COLLEGE

Mathematics Specialist Units 3,4
Test 2018

Section 1 Calculator Free

Systems of Equations, Vector Calculus

STUDENT'S NAME SOLUTIONS

DATE: Friday 18 May **TIME:** 20 minutes **MARKS:** 19

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

---

1. (4 marks)

Three planes are defined by the following equations:

$3x + 2y - z = 19$, $\underset{\sim}{\mathbf{r}} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 4$ and $2x + 4y - 5z = 32$. Determine the coordinates of the unique point of intersection of the three planes, using techniques of elimination.

$3x + 2y - z = 19$

$4x - y + 2z = 4$

$2x + 4y - 5z = 32$

$$\begin{bmatrix} 3 & 2 & -1 & 19 \\ 4 & -1 & 2 & 4 \\ 2 & 4 & -5 & 32 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & -1 & 19 \\ 11 & 0 & 3 & 27 \\ 18 & 0 & 3 & 48 \end{bmatrix} \begin{array}{l} \\ R1 + 2R2 \\ R3 + 4R2 \end{array}$$

$$\begin{bmatrix} 3 & 2 & -1 & 19 \\ 11 & 0 & 3 & 27 \\ 7 & 0 & 0 & 21 \end{bmatrix} \begin{array}{l} \\ \\ R3 - R2 \end{array}$$

$\therefore \ x = 3$

$z = -2$

$y = 4$

2. (8 marks)

Given $\underset{\sim}{\mathbf{r}}(t) = \left(3\sin\frac{t}{2}\right)\underset{\sim}{\mathbf{i}} + \left(2\cos\frac{t}{2}\right)\underset{\sim}{\mathbf{j}}$, where $\underset{\sim}{\mathbf{r}}(t)$ is the position vector of a particle at time t,

(a) Determine the cartesian equation of the path of the particle stating its shape. (3 marks)

$x = 3\sin\frac{t}{2}$ $\qquad$ $y = 2\cos\frac{t}{2}$

$\frac{x}{3} = \sin\frac{t}{2}$ $\qquad$ $\frac{y}{2} = \cos\frac{t}{2}$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

ELLIPSE

(b) Determine $\underset{\sim}{\mathbf{v}}(t)$ and $\underset{\sim}{\mathbf{a}}(t)$ (2 marks)

$v(t) = \frac{3}{2}\cos\frac{t}{2}\,i - \sin\frac{t}{2}\,j$

$a(t) = -\frac{3}{4}\sin\frac{t}{2}\,i - \frac{1}{2}\cos\frac{t}{2}\,j$

(c) Show that $\underset{\sim}{\mathbf{v}}(t)\cdot\underset{\sim}{\mathbf{a}}(t) = -\frac{5}{16}\sin t$ (3 marks)

$\begin{pmatrix} \frac{3}{2}\cos\frac{t}{2} \\ -\sin\frac{t}{2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{3}{4}\sin\frac{t}{2} \\ -\frac{1}{2}\cos\frac{t}{2} \end{pmatrix}$

$= -\frac{9}{8}\cos\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2}\cos\frac{t}{2}$

$= -\frac{5}{8}\cos\frac{t}{2}\sin\frac{t}{2}$

$= -\frac{5}{16} \times 2\sin\frac{t}{2}\cos\frac{t}{2}$

$= -\frac{5}{16}\sin t$

3. (7 marks)

Consider the following system of equations. Note: k is a constant.

$$\begin{aligned} x - 2y + 3z &= 1 \\ x + ky + 2z &= 2 \\ -2x + k^2y - 4z &= 3k - 4 \end{aligned}$$

(a) State the value(s) of k for which the system has an infinite number of solutions and give a geometric interpretation. (4 marks)

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 1 & k & 2 & 2 \\ -2 & k^2 & -4 & 3k-4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & k+2 & -1 & 1 \\ 0 & k^2+2k & 2 & 3k-2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & k+2 & -1 & 1 \\ 0 & k^2+2k & 0 & 3k \end{array}\right] \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array}$$

∞ SOLNS $\quad k^2+2k=0 \quad$ AND $\quad 3k=0$

$k(k+2)=0 \qquad k=0$

$k=0,-2$

$\therefore k=0$ ONLY SOLUTION

2 PLANES THE SAME AND INTERSECT THE OTHER PLANE AT A COMMON LINE

(b) State the value(s) of k for which the system has no solution. (1 mark)

$k^2+2k=0 \qquad 3k \neq 0$

$k \neq 0$

$\therefore k=-2$

(c) For what value(s) of k does the system have a unique solution? (2 marks)

$k^2+2k \neq 0$

$\therefore k \neq 0, -2$

TRINITY COLLEGE

Mathematics Specialist Units 3,4
Test 3 2018

Section 2 Calculator Assumed
Systems of Equations. Vector Calculus

STUDENT'S NAME ______________________________

DATE: Friday 18 May **TIME:** 35 minutes **MARKS**: 35

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser
Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

Intentional blank page

4. (13 marks)

The velocity vector $\underset{\sim}{v}(t)\ ms^{-1}$ of a particle is given by $\underset{\sim}{v}(t) = \left(\frac{3\pi}{4}\cos\frac{\pi t}{4}\right)\underset{\sim}{i} - \left(\frac{3\pi}{4}\sin\frac{\pi t}{4}\right)\underset{\sim}{j}$.

The position vector of the particle at time $t = 4$ is $\begin{pmatrix}0\\0\end{pmatrix}$.

(a) Determine, for any time t

(i) the displacement vector $\underset{\sim}{r}(t)$ (2 marks)

$r(t) = 3\sin\frac{\pi t}{4}\,i + 3\cos\frac{\pi t}{4}\,j + c$

$t=4 \quad \begin{pmatrix}0\\0\end{pmatrix} = 0i - 3j + c \qquad \therefore c = 3j$

$r(t) = 3\sin\frac{\pi t}{4}\,i + \left(3\cos\frac{\pi t}{4} + 3\right)j$

(ii) the speed $|\underset{\sim}{v}(t)|$ (1 mark)

$|v(t)| = \sqrt{\left(\frac{3\pi}{4}\cos\frac{\pi t}{4}\right)^2 + \left(\frac{3\pi}{4}\sin\frac{\pi t}{4}\right)^2}$

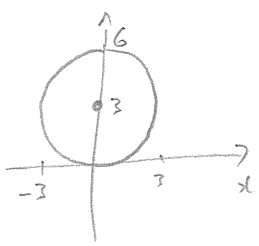
$= \frac{3\pi}{4}$ m/s

(iii) the acceleration $\underset{\sim}{a}(t)$ (1 mark)

$a(t) = -\frac{3\pi^2}{16}\sin\frac{\pi t}{4}\,i - \frac{3\pi^2}{16}\cos\frac{\pi t}{4}\,j$

(b) Describe the motion of the particle indicating the direction. (3 marks)

$x^2 + (y-3)^2 = 9$



$t=0 \quad \underset{\sim}{r} = \begin{pmatrix}0\\6\end{pmatrix}$

$t=1 \quad \underset{\sim}{r} = \begin{pmatrix}\frac{3}{\sqrt{2}}\\ \frac{3}{\sqrt{2}}+3\end{pmatrix}$

$\therefore$ CIRCULAR, CLOCKWISE

(c) Evaluate and interpret each of the following integrals.

(i) $\int_0^6 \underset{\sim}{\mathbf{v}}(t)\,dt = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$ (2 marks)

CHANGE IN DISPLACEMENT
$0 \le t \le 6$

(ii) $\int_0^6 |\underset{\sim}{\mathbf{v}}(t)|\,dt = \frac{9\pi}{2}$ (2 marks)

$\int_0^6$ NORM $(v(t))$

DISTANCE TRAVELLED IN THE
FIRST 6 SECONDS

(iii) $\left| \int_0^6 \underset{\sim}{\mathbf{v}}(t)\,dt \right| = \sqrt{18}$ (2 marks)

STRAIGHT LINE DISTANCE FROM $T=0$ TO $T=6$
or
CHANGE IN DISPLACEMENT $0 \le t \le 6$

5. (11 marks)

A point Q moving in the x-y plane has position vector $\underset{\sim}{\mathbf{r}}(t) = \begin{pmatrix} \cos t \\ 2\sin t \end{pmatrix}$. At $t = 0$ an insect crawls from the origin towards Q so that its position vector at time t is $\underset{\sim}{R}(t) = \underset{\sim}{r}(t) \times \sin t$, until it reaches Q, where it rests until $t = \dfrac{9\pi}{4}$ minutes.

(a) Determine the position vector of Q when $t = 0$ (1 mark)

$$r(t) = \begin{pmatrix} \cos 0 \\ 2\sin 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(b) How long does it take for the insect to first reach Q? (2 marks)

$$R(t) = Q(t)$$
$$\begin{pmatrix} \sin t \cos t \\ 2\sin^2 t \end{pmatrix} = \begin{pmatrix} \cos t \\ 2\sin t \end{pmatrix}$$

$\sin t \cos t = \cos t$ $\quad$ $2\sin^2 t = 2\sin t$

$t = 0, \frac{\pi}{2}$ $\quad$ $t = \frac{\pi}{2}$ $\quad$ $\therefore\ t = \frac{\pi}{2}$ MIN

(c) Show that $\underset{\sim}{R}(t) = \begin{pmatrix} \dfrac{\sin 2t}{2} \\ 1 - \cos 2t \end{pmatrix}$ (2 marks)

$\cos 2t = 1 - 2\sin^2 t$

$2\sin^2 t = 1 - \cos 2t$

$$\begin{pmatrix} \cos t \sin t \\ 2\sin t \sin t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \times 2\sin t \cos t \\ 2\sin^2 t \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}\sin 2t \\ 1 - \cos 2t \end{pmatrix}$$

(d) Determine the cartesian equation for the path of the insect before it reaches Q. (2 marks)

$x = \frac{1}{2}\sin 2t$ $\quad$ $y = 1 - \cos 2t$

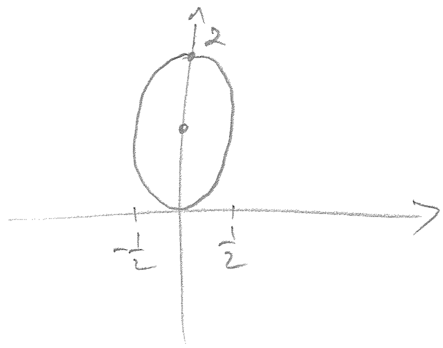
$2x = \sin 2t$ $\quad$ $1 - y = \cos 2t$

$$4x^2 + (1-y)^2 = 1$$

(e) Sketch the path of the insect indicating its direction. (2 marks)

$t = 0 \quad R(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$t = \frac{\pi}{2} \quad R(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$



ANTICLOCKWISE

(f) Determine the relationship between the velocity and acceleration of the insect at $t = \frac{\pi}{4}$ (2 marks)

$v(t) = (\cos 2t)i + (2\sin 2t)j$

$a(t) = (-2\sin 2t)i + (4\cos 2t)j$

$v\left(\frac{\pi}{4}\right) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad a\left(\frac{\pi}{4}\right) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$v \cdot a = 0$

$\therefore$ PERPENDICULAR

7. (10 marks)

A tennis ball is hit with an initial velocity of $\begin{pmatrix} 26 \cdot 5 \\ 2 \cdot 7 \end{pmatrix} ms^{-1}$ at a height of 60 cm above the ground and 6.4 m from the net. The net is 0.9 m high and the opponents half of the court is twelve metres in length.

(a) Determine the velocity vector and the position vector of the ball in terms of t (time) if the acceleration acting on the ball is given by $\boldsymbol{a} = \begin{pmatrix} 0 \\ -9 \cdot 8 \end{pmatrix} ms^{-2}$. (4 marks)

$v(t) = 26.5i + (2.7 - 9.8t)j$

$r(t) = 26.5i + (2.7t - 4.9t^2)j + (0i + 0.6j)$

$= 26.5i + (2.7t - 4.9t^2 + 0.6)j$

(b) Will the ball clear the net and if so by how much? (3 marks)

$26.5t = 6.4$

$t = 0.242$

$y = 0.966$

YES BY 0.066 m

(c) Will the ball land inside the opponent's half? Justify. (3 marks)

HT = 0

$-4.9t^2 - 2.7t + 0.6 = 0$

$t = 0.721$

$\therefore x_i = 19.08$

$> 6.4 + 12$

> 18.4

$\therefore$ LANDS OUT